18.06 FINAL

December 17, 2019 (180 minutes)

Please turn cell phones off completely and put them away.

No books, notes, or electronic devices are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Please write your name on **ALL** pages that you want graded (those will be the ones we scan).

The back sides of the paper will **NOT** be graded (for scratch work only).

Do not unstaple the exam, nor reorder the sheets.

If you need extra space for any problem, use the extra sheets at the back of the book. If you do so, please mention this in the main body of the problem in question, so we know to look.

There are 6 problems, worth 225 points in total.

NAME: ______________________________

MIT ID NUMBER: ______________________________

RECITATION INSTRUCTOR: ______________________________

PROBLEM 1

In this problem, consider the 4×4 matrix A whose columns are vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4 \in \mathbb{R}^4$:

$$A = \left[\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \mathbf{a}_3 \mid \mathbf{a}_4\ \right]$$

which are mutually orthogonal ($\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for all $i \neq j$) and have the following lengths:

$$||\mathbf{a}_1|| = 1, \quad ||\mathbf{a}_2|| = 3, \quad ||\mathbf{a}_3|| = 5, \quad ||\mathbf{a}_4|| = 7$$

(1) Compute $A^T A$. *(5 points)*

Solution: The entries of $A^T A$ just compute the pairwise inner products of the columns of A, which by the information provided in the problem are:

$$A^T A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 49 \end{bmatrix} = D$$

(2) Write A as a sum of four rank 1 matrices, and write each of those rank 1 matrices as a column vector times a row vector (the answer will depend on $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$). *(10 points)*

Solution: Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ be the standard basis vectors. With this notation we have:

$$A = \left[\ \mathbf{a}_1 \mid 0 \mid 0 \mid 0\ \right] + \left[\ 0 \mid \mathbf{a}_2 \mid 0 \mid 0\ \right] + \left[\ 0 \mid 0 \mid \mathbf{a}_3 \mid 0\ \right] + \left[\ 0 \mid 0 \mid 0 \mid \mathbf{a}_4\ \right] =$$

$$= \mathbf{a}_1\mathbf{e}_1^T + \mathbf{a}_2\mathbf{e}_2^T + \mathbf{a}_3\mathbf{e}_3^T + \mathbf{a}_4\mathbf{e}_4^T$$

(3) Compute the explicit formula for the solution $\mathbf{v} \in \mathbb{R}^4$ to the equation $A\mathbf{v} = \mathbf{b}$. Your answer should only depend on $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b} \in \mathbb{R}^4$. *Hint: remember (1).* *(10 points)*

Solution: Note that $A^T A$ is diagonal, so we just need to rescale the rows of A^T to compute the inverse. In other words, if we write $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 49 \end{bmatrix}$, then $A^{-1} = D^{-1}A^T$. Using this the solution to $A\mathbf{v} = \mathbf{b}$ is given by:

$$\mathbf{v} = A^{-1}\mathbf{b} = D^{-1}A^T\mathbf{b} = \begin{bmatrix} 1 \cdot \mathbf{a}_1^T\mathbf{b} \\ 9^{-1} \cdot \mathbf{a}_2^T\mathbf{b} \\ 25^{-1} \cdot \mathbf{a}_3^T\mathbf{b} \\ 49^{-1} \cdot \mathbf{a}_4^T\mathbf{b} \end{bmatrix}$$

For the explicit matrix A studied above, consider the matrix:

$$B = A \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(4) Write the third column of B in terms of the columns of A. *(5 points)*

Solution: The third column of B is given by the linear combination $-\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3$, described by the third column of the upper triangular matrix to the right of A.

(5) Compute the absolute value of the determinant of the matrix B from above. *(5 points)*

Solution: We first compute the determinant of A. To do this we use the equation $A^T A = D$. We also know that A and A^T have the same determinant. Then we get $1 * 9 * 25 * 49 = \det(D) = \det(A^T A) = \det(A)^2$, thus we get $|\det(A)| = 1 * 3 * 5 * 7 = 105$. Since the determinant of an upper triangular matrix is the product of the diagonal entries, we conclude that the determinant of B is given by:

$$|\det(B)| = |1 * 2 * (-1) * 3 \det(A)| = 6 * 105 = 630$$

(6) Which two columns of B are orthogonal? How do you know? *(5 points)*

Solution: The first column is a linear combination of $\mathbf{a}_1$ and the last column a linear combination of $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$. And since all the columns of A are orthogonal, we get the first and fourth column of B are orthogonal.

PROBLEM 2

Consider the matrix:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(1) Without doing any computations, what number is $\det C$, and why? *(5 points)*

Solution: The first two rows of C are the same, thus the matrix does not have full rank and so $\det(C) = 0$.

(2) Compute the eigenvalues λ_1, λ_2, λ_3 of C. Explain your reasoning. *(10 points)*

Solution: The characteristic polynomial is:

$$p(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}-\lambda \end{bmatrix} = -\lambda^3 + \frac{3\lambda^2}{2} - \frac{\lambda}{2} = -\lambda\left(\lambda^2 - \frac{3\lambda}{2} + \frac{1}{2}\right) = -\lambda(\lambda-1)\left(\lambda - \frac{1}{2}\right)$$

(you can get the formula for the determinant by, say, cofactor expansion along the first row, and the last equality by the quadratic formula). Thus we conclude that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = \frac{1}{2}$.

(3) Use the general method (i.e. Gauss-Jordan elimination to compute nullspaces) in order to compute eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of C corresponding to the eigenvalues λ_1, λ_2, λ_3.

(10 points)

Solution: We compute the eigenvector of $\lambda_1 = 1$, so we compute the eigenvector by computing the nullspace of $C - I$, so we get

$$C - I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the second column is the only free column and so we get the eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

We compute the eigenvector of $\lambda_2 = 0$, so we compute the eigenvector by computing the nullspace of C, so we get

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the third column is the only free column and so we get the eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

We compute the eigenvector of $\lambda_3 = \frac{1}{2}$, so we compute the eigenvector by computing the nullspace of $C - \frac{1}{2}I$, so we get

$$C - \frac{1}{2}I = \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ 1 & -\frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the third column is the only free column and so we get the eigenvector is $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

(4) For the given matrix C and its eigenvalues λ_1, λ_2, λ_3 computed in part (2), calculate:

$$(C - \lambda_1 \cdot I)(C - \lambda_2 \cdot I)(C - \lambda_3 \cdot I)$$

(5 points)

Solution: We have:

$$(C - 1 \cdot I)(C)(C - \frac{1}{2} \cdot I) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ 1 & -\frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ 1 & -\frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} = 0$$

(5) Use part (4) to get formulas for C^3 <u>and</u> C^4 in terms of C^2 and C only (i.e. your formulas should be of the form $C^3 = xC^2 + yC$ for some constants x, y, and similarly for C^4).

(5 points)

Solution: By expanding the equality in the previous part, we have:

$$C^3 - \frac{3}{2} \cdot C^2 + \frac{1}{2} \cdot C = 0 \quad \Rightarrow \quad C^3 = \frac{3}{2} \cdot C^2 - \frac{1}{2} \cdot C \tag{1}$$

This takes care of C^3. As for C^4, let's just multiply the formula above by C:

$$C^4 = \frac{3}{2} \cdot C^3 - \frac{1}{2} \cdot C^2$$

However, we need an answer in terms of C^2 and C only, so let's replace C^3 by the expression (1). We obtain:

$$C^4 = \frac{7}{4} \cdot C^2 - \frac{3}{4} \cdot C$$

(6) Consider any vector of the form $\mathbf{v} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3$, where α, β, γ are real numbers, and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the eigenvectors of C computed in part (3). Compute:

$$C^n \mathbf{v}$$

in terms of α, β, γ, n and vectors with explicit numbers as entries.

Then find and prove an explicit formula for:

$$\lim_{n \to \infty} C^n \mathbf{v}$$

in terms of α, β, γ and vectors with explicit numbers as entries. *(5 points)*

Solution: Because $C\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all $i \in \{1, 2, 3\}$, we have $C^n \mathbf{v}_i = \lambda_i^n \mathbf{v}_i$. Plugging in the explicit values for the λ's and the $\mathbf{v}$'s, we have:

$$C^n \mathbf{v} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{\gamma}{2^n} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Note that the first summand does not change as $n \to \infty$, while the second summand goes to 0. Thus we get

$$\lim_{n\to\infty} C^n \mathbf{v} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

PROBLEM 3

Let $M = \begin{bmatrix} 1 & -1 & 3 & z \\ 0 & -1 & y & -1 \\ x & 2 & -1 & 5 \end{bmatrix}$ and consider the equation (with $\mathbf{v} \in \mathbb{R}^4$ and $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \in \mathbb{R}^3$):

$$M\mathbf{v} = \mathbf{n}$$

We know that the equation has solutions $\mathbf{v} \in \mathbb{R}^4$ if and only if $n_1 + n_2 + n_3 = 0$.

(1) Based on the information given in the previous sentence, what is the column space of M and what are the mystery numbers x, y, z? Explain how you know. *(5 points)*

Solution: The above description of when the equation $M\mathbf{v} = \mathbf{n}$ has a solution is exactly the description of the column space. Ie the column space is given by vectors $\mathbf{n}$ such that the entries add up to 0. Therefore, all the columns of A must also have the property that their entries add up to 0. We conclude:

$$x = -1$$
$$y = -2$$
$$z = -4$$

(2) For which values of $\mathbf{n}$ is the set $\{\mathbf{v} \in \mathbb{R}^4 \text{ such that } M\mathbf{v} = \mathbf{n}\}$ a subspace of $\mathbb{R}^4$, and why? *(5 points)*

Solution: Note that 0 is an element of every subspace of $\mathbb{R}^4$. Thus if:

$$0 \in \{\mathbf{v} \in \mathbb{R}^4 \text{ such that } M\mathbf{v} = \mathbf{n}\}$$

we must have $\mathbf{n} = M0 = 0$, so we must have $\mathbf{n} = 0$. In fact for $\mathbf{n} = 0$ the set in question is indeed a subspace. To check this we need to see that if $x, y \in \{\mathbf{v} \in \mathbb{R}^4 \text{ such that } M\mathbf{v} = \mathbf{n}\}$ and $\lambda \in \mathbb{R}$, then we know $Mx = My = 0$. Then we have $M(x + y) = M(\lambda x) = 0$ and thus $x + y, \lambda x \in \{\mathbf{v} \in \mathbb{R}^4 \text{ such that } M\mathbf{v} = \mathbf{n}\}$. Thus we get this is indeed a subspace exactly when $\mathbf{n} = 0$.

(3) With the specific values you found for x, y, z, find the complete solution to the equation:

$$M\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(solve this part by Gauss-Jordan elimination, identifying free/pivot variables and all that). *(10 points)*

Solution: We do this by using Gaussian elimination with the extended matrix to get

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & -4 & 0 \\ 0 & -1 & -2 & -1 & 1 \\ -1 & 2 & -1 & 5 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 3 & -4 & 0 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 1 & 2 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 5 & -3 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus here we get a particular solution is given by $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and the solution to $M\mathbf{v} = 0$ is a subspace with basis given by $\begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ (one for each of the two free columns, namely the third and fourth columns above). Thus the general solution is:

$$\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

for any constants α and β.

(4) Let $V \subset \mathbb{R}^3$ be the orthogonal complement of the column space you computed in part (1).

- V is called the left nullspace of M. *(3 points)*
- Compute the projection matrix onto V. *(7 points)*

Solution: The left nullspace and the column space are exactly orthogonal complements to each other. Since the column space precisely consists of those vectors whose entries sum up to 0, then its orthogonal complement is precisely the line spanned by the vector:

$$c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the projection matrix if given by

$$P_V = \frac{cc^T}{c^Tc} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(5) If N is any other matrix such that the column space of N is contained inside the nullspace of M, then what is the product MN equal to? Justify your answer. *(5 points)*

Solution: By assumption, any column $\mathbf{v}$ of N has the property that $M\mathbf{v} = 0$. Therefore, we conclude that $MN = 0$.

PROBLEM 4

(1) Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}$. Compute $A^T A$, its eigenvalues and eigenvectors. *(10 points)*

Solution: We have

$$A^T A = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$$

The eigenvalues and eigenvectors are given by:

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \lambda = 25, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The way to do this is by analogy with parts (2) and (3) of Problem 2; while we expect you to compute the eigenvalues by solving the characteristic polynomial, in this problem only we won't take off points for simply guessing the eigenvectors. The normalization of the eigenvectors by the factor $\sqrt{2}$ is arbitrary at this point, but it will be important in the next part.

(2) Compute the SVD of the matrix A from part (1), i.e. write it as:

$$A = U \Sigma V^T$$

where $U^T U = I_4$, $V^T V = I_2$, and the only non-zero entries of Σ are on its diagonal. You should deduce Σ and V from your work in part (1). Afterwards, this information will give you two columns of U, but you <u>must compute the other two</u> by Gram-Schmidt. *(10 points)*

Solution: The singular values are given by the square roots of the eigenvalues of $A^T A$ and so are given by $\sigma_1 = 1$ and $\sigma_2 = 5$. The columns of V are given by the above eigenvectors. Thus

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The first two columns of U are given by the following equations $u_i = \frac{Av_i}{\sigma_i}$. Thus we get

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_2 = \frac{1}{5\sqrt{2}} \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix}$$

To find the other columns of U we need to extend the above set to an orthonormal set. The way to do so is to consider an arbitrary basis u_1, u_2, a, b where $a, b \in \mathbb{R}^4$ are some simple vectors (e.g. coordinate unit vectors), and apply the Gram-Schmidt process. This process will transform a, b into the vectors:

$$u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad u_4 = \frac{1}{5\sqrt{2}} \begin{bmatrix} 4 \\ -3 \\ -3 \\ 4 \end{bmatrix}$$

such that now u_1, u_2, a, b form an orthonormal basis. Thus we get

$$A = \frac{1}{5\sqrt{2}} \begin{bmatrix} 5 & 3 & 0 & 4 \\ 0 & 4 & 5 & -3 \\ 0 & 4 & -5 & -3 \\ -5 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(3) Let $S = A^T A$ be the symmetric 2×2 matrix you computed in part (1), and define:

$$e(x, y) = \begin{bmatrix} x & y \end{bmatrix} S \begin{bmatrix} x \\ y \end{bmatrix}$$

- The explicit formula for the quantity above is $e(x, y) = \underline{13x^2 + 24xy + 13y^2}$ *(3 points)*
- We have $e(x, y) > 0$ for all $(x, y) \neq (0, 0)$ because S is <u>positive definite</u> *(3 points)*
- $\{e(x, y) = 1\}$ is the equation of the following geometric shape in the x, y plane:

<u>ellipse</u>

(3 points)

(4) Obtain a linear transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ which transforms the geometric shape $\{e(x, y) = 1\}$ into a circle centered at the origin (full points for a correct formula for ϕ with explanation of how you got it, half points for a good geometric description of ϕ in words). *(6 points)*

Solution: We have to use a linear transformation that maps both axe of the elipse to the standard basis. To do this we start with the transpose of matrix of unit eigenvectors V^T and then rescale each eigenvector by the square root of the corrsponding eigenvalue. Thus we get that the corresponding linear transformation is given by the matrix.

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 5 & 5 \end{bmatrix}$$

PROBLEM 5

(1) Compute the determinant of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Show all the steps. *(10 pts)*

Solution: To do this we will proceed by elimination. Thus we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Here we did one swap of rows, thus we get

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} = -(1 * (-1) * (-1) * (-1) * 4) = 4$$

Here we use that the determinant of an upper triangular matrix is given by the product of the diagonal entries.

(2) Compute $\det \begin{bmatrix} 0 & x & 0 & -1 \\ 2 & -5 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix}$ as a function of x. Show your steps. *(10 points)*

Solution: We compute this determinant by using cofactor expansion on the first row:

$$\det \begin{bmatrix} 0 & x & 0 & -1 \\ 2 & -5 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix} = -x \cdot \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} - (-1) \cdot \det \begin{bmatrix} 2 & -5 & 0 \\ 0 & 4 & -1 \\ 3 & 0 & -2 \end{bmatrix} =$$

$$= 2x + (2 * 4 * (-2) + (-5) * (-1) * 3) = 2x - 1$$

(3) Pick a complex (non-real) number z and write it in both Cartesian ($z = a + bi$) and polar ($z = re^{i\theta}$) form. In doing so, give the formula which connects r, θ to a, b. *(5 points)*

Solution: The formula that goes from r, θ to a, b is:

$$r = \sqrt{a^2 + b^2} \qquad \theta = \arccos\left(\frac{a}{\sqrt{a^2 + b^2}}\right)$$

So you could take, for instance, $z = i = 0 + i \cdot 1$. Then by the formula above, we have $r = 1$ and $\theta = \frac{\pi}{2}$. Therefore:

$$z = e^{\frac{i\pi}{2}}$$

(4) For the z you just chose, compute $z + \bar{z}$ and $z\bar{z}$, where $\bar{z} = a - bi$ is the conjugate of z. Write down a 2×2 matrix J with real entries, whose eigenvalues are z and $\bar{z}$. *(5 points)*

Solution: For $z = i$, we have $z + \bar{z} = 0$ and $z\bar{z} = 1$. For a matrix to have eigenvalues z and $\bar{z}$ we thus need the trace to be 0 and the determinant should be 1. Thus we can take

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

As this matrix has indeed the correct trace and determinant and thus it has the correct characteristic polynomial and so the correct eigenvalues.

(5) Consider the solution to the system of differential equations:

$$\begin{bmatrix} \dot{f}(t) \\ \dot{g}(t) \end{bmatrix} = J \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

where J is the matrix you chose in part (4). Any solution to this equation is of the form:

$$f(t) = c_1 \cdot \ldots\ldots\ldots\ldots\ldots\ldots \;+\; c_2 \cdot \ldots\ldots\ldots\ldots\ldots\ldots$$

where c_1 and c_2 are some scalars. Fill the blanks with two explicit functions of t, which you should get from what you already know about the matrix J (no further computations are necessary, but you should say explicitly what these functions have to do with J). *(5 points)*

Solution: We know the general solution is given by:

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = e^{Jt}\mathbf{a}$$

for some constant vector $\mathbf{a}$. To take the exponential, we need to diagonalize J:

$$J = V \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} V^{-1} \quad \Rightarrow \quad e^{Jt} = V \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} V^{-1}$$

for some matrix V whose entries are numbers. Therefore, the solution to the differential equation is:

$$\begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = V \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} V^{-1}\mathbf{a}$$

Since the entries of V and $\mathbf{a}$ are numbers (which do not depend on t), we conclude that:

$$f(t) = c_1 \cdot e^{it} + c_2 \cdot e^{-it} \tag{2}$$

for various numbers c_1 and c_2.

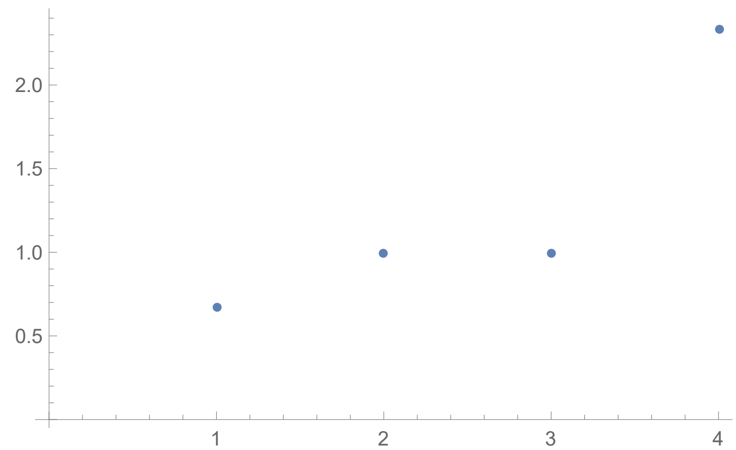
(6) Explain (in words) the qualitative behavior of $f(t)$ from part (5) as $t \to \infty$. *(5 points)*

Solution: Since $e^{\pm it} = \cos(t) + i\sin(t)$, as $t \to \infty$ the function (2) oscillates just like the functions sine and cosine. Note that if you would have chosen $z = a + ib$ for some general real numbers a and b, then the function $f(t)$ would have:

- tended to ∞ as $t \to \infty$ if $a > 0$
- tended to 0 as $t \to \infty$ if $a < 0$

PROBLEM 6

At times 1, 2, 3, 4, you measure temperatures $\frac{2}{3}$, 1, 1, $\frac{7}{3}$, respectively. These are represented as data points on the following plot (horizontal axis is time, vertical axis is temperature):



The goal is to find numbers a, b such that the line $y = ax + b$ is the best fit for the data points above. Specifically, this means that a and b should be chosen such that the quantity:

$$\Upsilon = \left(a \cdot 1 + b - \frac{2}{3}\right)^2 + (a \cdot 2 + b - 1)^2 + (a \cdot 3 + b - 1)^2 + \left(a \cdot 4 + b - \frac{7}{3}\right)^2 \quad \text{is minimal}$$

(1) Define a 4×2 matrix A, a 2×1 vector $\mathbf{v}$, and a 4×1 vector $\mathbf{b}$ such that:

$$\Upsilon = ||A\mathbf{v} - \mathbf{b}||^2$$

(the entries of A and $\mathbf{b}$ should be numbers, and those of $\mathbf{v}$ should be unknowns). *(5 points)*

Solution: Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \\ \frac{7}{3} \end{bmatrix}$$

With this matrix and vectors, we indeed get $\Upsilon = ||A\mathbf{v} - \mathbf{b}||^2$.

(2) Write down the general formula for the solution to the least squares problem: if the columns of the matrix A are independent, then the quantity $||A\mathbf{v} - \mathbf{b}||$ is minimal for:

$$\mathbf{v} = \underline{(A^T A)^{-1} A^T \mathbf{b}}$$

(the answer should be a formula in terms of A and $\mathbf{b}$; don't plug in numbers yet). *(5 points)*

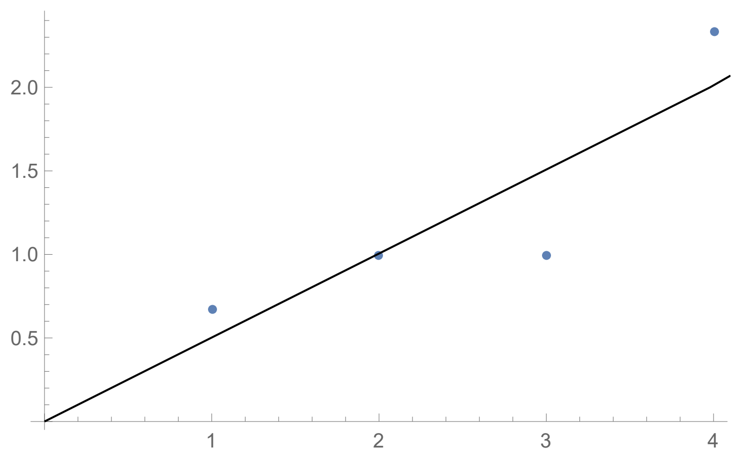
(3) Use the previous two parts to solve for the numbers a and b that give the best line fit for our four data points (i.e. that minimize the quantity Υ). Show your work. *(10 points)*

Solution: From the formula we get

$$\mathbf{v} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 5 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} 15 \\ 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence we conclude that the best fit line is $y = \frac{x}{2}$.

(4) With the values you just found for a and b, draw the precise line $y = ax + b$ in the graph:



(5 points)

Let's look at the aforementioned data points from the point of view of statistics. Put the sample sets for time and temperature in a 4×2 matrix:

$$\mathbf{Z} = \left[\ \mathbf{x} \mid \mathbf{y}\ \right] \qquad \text{where } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \\ \frac{7}{3} \end{bmatrix}$$

The covariance matrix of the sample sets $\mathbf{x}$ = "time" and $\mathbf{y}$ = "temperature" is given by:

$$K = \begin{bmatrix} K_{\mathbf{xx}} & K_{\mathbf{xy}} \\ K_{\mathbf{yx}} & K_{\mathbf{yy}} \end{bmatrix} = \frac{\mathbf{Z}^T P \mathbf{Z}}{4-1}, \qquad \text{where } P = \frac{1}{4} \cdot \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

A computer tells us that this matrix takes the form (up to two decimals approximation):

$$K = \underbrace{\begin{bmatrix} -0.53 & 1.87 \\ 1 & 1 \end{bmatrix}}_{\text{orthogonal}} \begin{bmatrix} 0.1 & 0 \\ 0 & 2.11 \end{bmatrix} \begin{bmatrix} -0.53 & 1 \\ 1.87 & 1 \end{bmatrix}$$

(5) Using all the information above, for any constants α and β, write the variance of the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ in terms of the matrix K and the vector $\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$ *(5 points)*

Solution: The variance of $\alpha\mathbf{x} + \beta\mathbf{y}$ is:

$$K_{\alpha\mathbf{x}+\beta\mathbf{y},\alpha\mathbf{x}+\beta\mathbf{y}} = \alpha^2 K_{\mathbf{xx}} + \alpha\beta K_{\mathbf{xy}} + \beta\alpha K_{\mathbf{yx}} + \beta^2 K_{\mathbf{yy}} = \begin{bmatrix}\alpha & \beta\end{bmatrix} K \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

(6) Using all the information above, find particular constants α and β such that the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$ has variance precisely 0.1 (and explain how you know) *(5 points)*

Solution: We are told that:

$$K = Q \begin{bmatrix}0.1 & 0 \\ 0 & 2.11\end{bmatrix} Q^T$$

for some explicit matrices, where Q is orthogonal. Therefore, we conclude that:

$$K_{\alpha\mathbf{x}+\beta\mathbf{y},\alpha\mathbf{x}+\beta\mathbf{y}} = \mathbf{c}^T Q \begin{bmatrix}0.1 & 0 \\ 0 & 2.11\end{bmatrix} Q^T \mathbf{c}$$

where $\mathbf{c} = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$. If you want the quantity above to be 0.1, you may choose $\mathbf{c}$ such that:

$$Q^T \mathbf{c} = \begin{bmatrix}1 \\ 0\end{bmatrix} \quad \Rightarrow \quad \mathbf{c} = Q \begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}-0.53 \\ 1\end{bmatrix}$$